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LONGITUDINAL LAMINAR FLOW IN AN ARRAY OF CIRCULAR CYLINDERS

J. SCHMID

The Nuclear Research Institute of Czechoslovak Academy of Sciences, Řež by Prague, Czechoslovakia

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Abstract—The paper describes a theoretical investigation of longitudinal, developed laminar flow in an infinite square array of circular cylinders, the array being bounded on one side by a wall, parallel to the cylinder axes. The influence of the array pitch ($2b$) and the distance of the wall from the first row of cylinders (b_1) on the flow rate of the medium through the individual cells of the array is studied and the mutual influence of the flow rate in individual cells one upon the others is investigated. The quantity b has been varied in the range from $1{\cdot}5\ R$ to $2{\cdot}5\ R$ (R being the cylinder radius) and the distance from the wall has been varied from $b_1 = b$ (the first cell being square) to such values of b_1, where the ratio of hydraulic diameters of the first cell and a cell within the lattice ($\xi = d_1/d_i$) reaches a value of 1·3. From the calculation it follows that the mutual influence of the individual cells is relatively small. Variations of the flow rate in the first two rows of cells reach a value of 1 per cent only for relatively large pitches ($b > 2{\cdot}5\ R$) and for cases, where the first cell differs greatly in the hydraulic diameter from the cells inside the array ($\xi > 1{\cdot}3$ and $\xi < 0{\cdot}6$). The influence of the third row of cells is negligible in the range of parameters considered.

NOMENCLATURE

- r, radius vector [m];
- φ, azimuth angle [rad];
- R, cylinder radius [m];
- ρ, dimensionless radius vector, $\rho = r/R$;
- $2b$, array pitch [m];
- $2s$, dimensionless array pitch, $s = b/R$;
- b_1, distance of the wall from the first row of cells [m];
- s_1, dimensionless distance of the wall from the first row of cells, $s_1 = b_1/R$;
- d_i, dimensionless hydraulic diameter of a cell within the array according to relation (34);
- d_1, dimensionless hydraulic diameter of the first cell according to relation (33);
- $w_z(r, \varphi)$, velocity of the liquid [m/s];
- $W(\rho, \varphi)$, dimensionless velocity according to relation (2);
- G, dimensionless volume flow rate according to relation (30);
- ξ, ratio of hydraulic diameters, $\xi = d_1/d_i$;
- η, ratio of dimensionless volume flow rates, $\eta = G/G_1$ or $\eta = G/G_i$;
- J, total number of considered cells;
- j, number of the cell.

1. INTRODUCTION

IN THERMAL calculations of heat exchangers and in some types of nuclear reactor fuel elements we find the problem of determining the flow rate of the medium for the case of longitudinal flow in a bundle of tubes (or rods).

In actual calculations, the procedure generally applied is to divide the bundle into a number of "cells" more or less by estimation (see Fig. 1, the shaded area). The cell thus selected is considered to be "isolated" from the other cells, i.e. we assume that the flow of the medium through this cell is not influenced by the neighbouring cells. Based on the hydraulic diameter of this cell, and on other quantities (pressure gradients, etc.) the flow rate of the medium through the cell considered is calculated. In the case of a regular and infinite array, the procedure described above is certainly correct. If, however, the bundle is limited by a fixed wall,

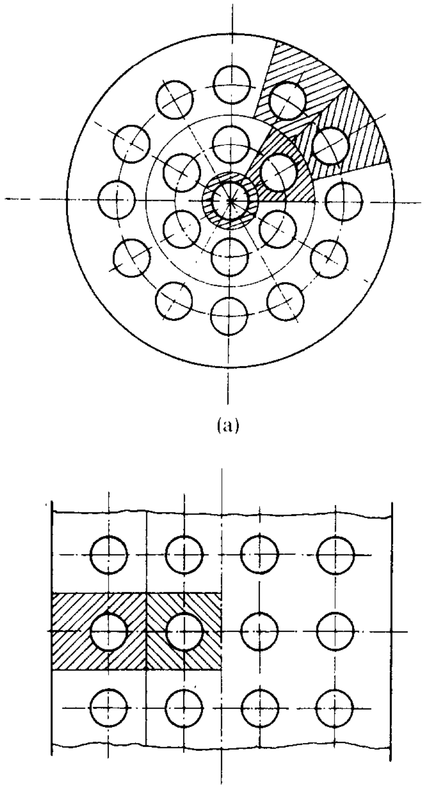


FIG. 1. Scheme of a heat exchanger or fuel element.

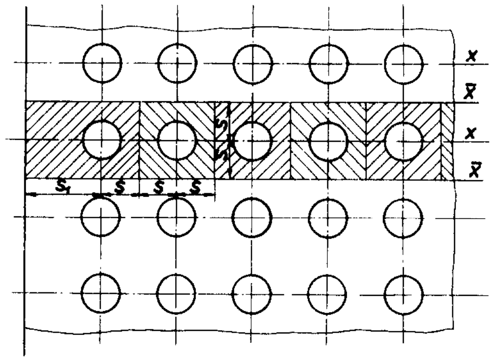


FIG. 2. Schematic illustration of the problem studied.

such a method may be assumed to be only an approximation.

The object of this paper is to determine the number of cell rows which will be influenced by the wall, from the point of view of the flow rate of the medium through these cells. To illustrate the influence of the wall on flow in the rod bundle, the laminar flow of a liquid through a bundle of rods is solved in this paper in the "semi-infinite geometry", limited by a fixed wall. The rods are arranged in a square array with constant pitch.

2. BOUNDARY CONDITIONS

To solve the problem, we divide the cross-section of the bundle into elementary cells (Fig. 2). In the case given, where we shall have regard to the mutual influence of the cells, it is virtually immaterial how we select the cells. A typical cell is shown in Fig. 3.

We shall proceed to solve the problem by determining an expression for the velocity field in the vicinity of every circular cylinder (i.e. in every cell), and using boundary conditions for the boundary of individual cells we shall match (sew together) these partial velocity fields.

Since the array is semi-infinite, the solution is symmetrical with respect to x-axis (see Fig. 2). It is therefore sufficient to investigate one row of cells, shown in Fig. 2 by the shaded area. Because in the calculation we can only consider a finite number of cells, we shall limit the last cell by the boundary condition in the form of $\partial w_z/\partial x|_{x=b} = 0$.

We are therefore going to study the problem as it is shown in Fig. 1(b). We shall gradually increase the number of cells, until the results for the first cells vary no more.

The longitudinal laminar flow parallel to the z-axis is described by a partial differential equation in the form of*

$$\mu \nabla^2 w_z = \frac{dp}{dz}. \qquad (1)$$

where μ is the dynamic viscosity and dp/dz

* In the given example, two Navier–Stokes equations are eliminated ($w_x \equiv 0$, $w_y \equiv 0$) and the third is reduced to the Poisson equation.

the pressure gradient. By means of the substitution

$$W = w_z \left(-\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z}R^2\right)^{-1} \qquad \rho = \frac{r}{R} \tag{2}$$

we transform equation (1) into the dimensionless form

$$\nabla^2 W = -1. \tag{3}$$

In our case equation (3) has the form

$$\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \varphi^2} = -1. \tag{4}$$

The solution of this partial differential equation takes the form

$$W = A_0 + B_0 \ln \rho - (\rho^2/4) + \sum_{n=1}^{\infty} [(A_n\rho^n + B_n\rho^{-n}) \cos n\varphi + (C_n\rho^n + D_n\rho^{-n}) \sin n\varphi]. \tag{5}$$

Since the problem is symmetrical with respect to the x-axis, the part of the solution which contains the sin $n\varphi$ functions is eliminated.

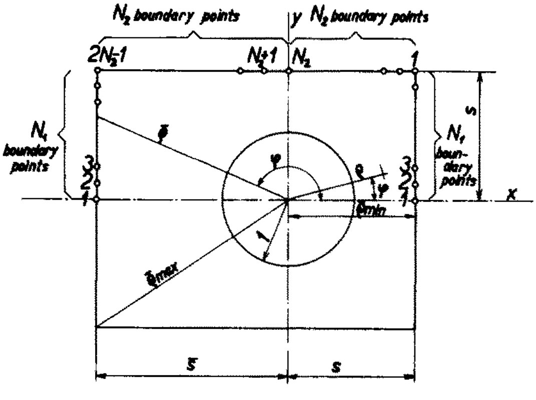


FIG. 3. Notation of the cell dimensions and location of boundary points.

One half of the remaining integration constants may be eliminated due to the boundary condition for the radius $r = R$, i.e. $\rho = 1$, where the relation holds:

$$W(1, \varphi) = 0. \tag{6}$$

Based on the series (5) we obtain the following relations for the integration constants:

$$A_0 = \tfrac{1}{4}$$

$$A_n = -B_n \qquad (n = 1, 2, 3, \ldots).$$

Therefore, the expression for the dimensionless velocity will take the form

$$W(\rho, \varphi) = \tfrac{1}{4}(1 - \rho^2) + B_0 \ln \rho + \sum_{n=1}^{\infty} B_n(\rho^n - \rho^{-n}) \cos n\varphi. \tag{7}$$

2.1. *Boundary conditions for the cell*

In the given case, where the solution of equation (3) is described in polar coordinates ρ, φ, the boundary condition of the form (6) is satisfied exactly along the whole circumference of the circle $\rho = 1$. The situation is somewhat different on the boundary of a cell of rectangular or square shape. Papers [1, 2] point out that the boundary conditions on the cell surface cannot be satisfied precisely for the case of a finite number of terms of the expansion. It can, however, be required that the boundary conditions be satisfied in selected points of the cell boundary.

The dimensions of a cell and the distribution of boundary points along the circumference is shown in Fig. 3. For the dimension $\bar{s}$, the relation holds:

$\bar{s} = s_1$ for the first cell

$\bar{s} = s$ for the other cells of the bundle.

In the interval $0 \leqslant y \leqslant s$ we locate N_1 points distributed regularly, and in the interval $-s \leqslant x \leqslant 0$ and $0 \leqslant x \leqslant s$ we locate N_2 points. Let us give the boundary conditions for typical cells, and determine the dependence between the selected number of points N_1 and N_2 and the number of terms of the expansion N_3 (7).

On the boundary $x = \pm s$ of the individual cells, the condition of continuity of the velocity

W and the velocity gradient must be satisfied, i.e.

$$W_j|_{x=s} = W_{j+1}|_{x=-s} \qquad \text{for the points } 1, 2, \ldots, N_1, j = 1, 2, \ldots, J-1; \tag{8a}$$

$$\left.\frac{\partial W_j}{\partial x}\right|_{x=s} = \left.\frac{\partial W_{j+1}}{\partial x}\right|_{x=-s} \qquad \text{for the points } 1, 2, \ldots, N_1, j = 1, 2, \ldots, J-1; \tag{8b}$$

$$\left.\frac{\partial W_j}{\partial y}\right|_{x=s} = \left.\frac{\partial W_{j+1}}{\partial y}\right|_{x=-s} \qquad \text{for the points } 2, 3, \ldots, N_1 - 1, j = 1, 2, \ldots, J-1; \tag{8c}$$

where J is the total number of cells considered.

For point No. 1 the condition (8c) is already satisfied, due to the symmetry of the solution with respect to the x-axis.

On the boundary $y = s$, as a consequence of the symmetry of the solution (around the $\bar{x}$-axis, Fig. 2) the following boundary condition holds

$$\left.\frac{\partial W_j}{\partial y}\right|_{y=s} = 0 \qquad \text{for the points } 1, 2, \ldots, 2N_2 - 1, j = 1, 2, \ldots, J. \tag{9}$$

For the left boundary ($x = -\bar{s}$) of the first cell,

$$W_1|_{x=-\bar{s}} = 0 \qquad \text{for points } 1, 2, \ldots, N_1, \tag{10a}$$

$$\left.\frac{\partial W_1}{\partial y}\right|_{x=-\bar{s}} = 0 \qquad \text{for points } 2, 3, \ldots, N_1 - 1. \tag{10b}$$

On the right boundary ($x = s$) of the last cell (denoted J) it is possible, as already mentioned, to write the condition

$$\frac{\partial W_J}{\partial x}_{x=s} = 0 \qquad \text{for points } 1, 2, \ldots, N_1. \tag{11}$$

The total number of boundary conditions for J cells is given by the relation

$$K = J(3N_1 + 2N_2 - 3). \tag{12}$$

Therefore, from expression (12) it follows that in every cell we must select

$$N_3 = 3N_1 + 2N_2 - 3 \tag{13}$$

terms in the expansion of form (7).

The boundary conditions for a single cell are given by conditions (9), (10a), (10b), (11). The total number of conditions agrees with the given value of N_3.

2.2. *Expressions for the velocity W and its derivates*

The series (7) for the dimensionless velocity may be re-written in the form

$$W = b + \sum_{n=0}^{N_3-1} B_n a_n, \tag{14}$$

where

$$\left.\begin{aligned} b &= \tfrac{1}{4}(1-\rho^2) \\ a_0 &= \ln \rho \\ a_n &= \chi_n \cos n\varphi \qquad (n = 1, 2, \ldots, N_3 - 1) \end{aligned}\right\} \tag{15}$$

$$\chi_n = \rho^n - \rho^{-n}. \tag{16}$$

We obtain the expressions for the derivates $\partial W/\partial x$ and $\partial W/\partial y$ by substituting for $\partial W/\partial \rho$ and $\partial W/\partial \varphi$ into the expressions

$$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial \rho}\cos\varphi - \frac{1}{\rho}\frac{\partial W}{\partial \varphi}\sin\varphi,$$

$$\frac{\partial W}{\partial y} = \frac{\partial W}{\partial \rho}\sin\varphi + \frac{1}{\rho}\frac{\partial W}{\partial \varphi}\cos\varphi.$$

It can be deduced, that

$$\frac{\partial W}{\partial x} = d + \sum_{n=0}^{N_3-1} B_n c_n, \tag{17}$$

$$\frac{\partial W}{\partial y} = g + \sum_{n=0}^{N_3-1} B_n f_n, \tag{18}$$

where

$$d = -\frac{\rho}{2}\cos\varphi, \qquad g = -\tfrac{1}{2}\rho\sin\varphi,$$

$$c_0 = \frac{1}{\rho}\cos\varphi, \qquad f_0 = \frac{1}{\rho}\sin\varphi,$$

$$\left.\begin{aligned} c_n &= \alpha_n\cos\varphi + \beta_n\sin\varphi \\ f_n &= \alpha_n\sin\varphi - \beta_n\cos\varphi \end{aligned}\right\} n = 1, 2, \ldots, N_3 - 1, \tag{19}$$

and where

$$\alpha_n = \frac{n\psi_n}{\rho}\cos n\varphi,$$

$$\beta_n = \frac{n\chi_n}{\rho}\sin n\varphi,$$

$$\psi_n = \rho^n + \rho^{-n},$$

$$\chi_n = \rho^n - \rho^{-n}.$$

2.3. *The final notation of the boundary conditions*

It has been found by analysis of the boundary conditions and evaluation of the partial numerical results, that the boundary conditions (8c), (10b) in the form of

$$\left.\frac{\partial W}{\partial y}\right|_{x\pm s} = 0 \qquad \text{for the points } 2, 3, \ldots, N_1 - 1$$

have little influence on the solution. They have therefore been left out of the calculation. For the total number of terms of the expansion (7) we obtain the expression:

$$N_3 = 2(N_1 + N_2) - 1. \tag{20}$$

For the sake of clarity and the possibility of easier programming for a digital computer we now rewrite the boundary conditions in the matrix form.

We introduce the matrixes $\mathbf{M}_j$ and $\overline{\mathbf{M}}_{j+1}$ and vectors $\mathbf{B}_j$, $\mathbf{C}_j$ and $\overline{\mathbf{C}}_{j+1}$ in the form of

$$\mathbf{M}_j = \begin{bmatrix} a_{10}^{(j)}, & a_{11}, & a_{12}, \ldots, & a_{1K} \\ a_{20}, & a_{21}, & a_{22}, \ldots, & a_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N_10}, & a_{N_11}, & a_{N_12}, \ldots, & a_{N_1K} \\ c_{10}, & c_{11}, & c_{12}, \ldots, & c_{1K} \\ c_{20}, & c_{21}, & c_{22}, \ldots, & c_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N_10}, & c_{N_11}, & c_{N_12}, \ldots, & c_{N_1K} \end{bmatrix} \quad \overline{\mathbf{M}}_{j+1} = \begin{bmatrix} a_{10}^{(j+1)}, & a_{11}, & a_{12}, \ldots, & a_{1K} \\ a_{20}, & a_{21}, & a_{22}, \ldots, & a_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N_10}, & a_{N_11}, & a_{N_12}, \ldots, & a_{N_1K} \\ c_{10}, & c_{11}, & c_{12}, \ldots, & c_{1K} \\ c_{20}, & c_{21}, & c_{22}, \ldots, & c_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ c_{N_10}, & c_{N_11}, & c_{N_12}, \ldots, & c_{N_1K} \end{bmatrix} \tag{21}$$

$$\mathbf{B}_j = \begin{bmatrix} B_0^{(j)} \\ B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B_k \end{bmatrix}, \qquad \mathbf{C}_j = \begin{bmatrix} b_1^{(j)} \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{N_1} \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_{N_1} \end{bmatrix}, \qquad \overline{\mathbf{C}}_{j+1} = \begin{bmatrix} b_1^{(j+1)} \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{N_1} \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_{N_1} \end{bmatrix}, \tag{22}$$

where $K = N_3 - 1$. Indexes j and $j + 1$ in the matrixes and vectors have been left out for simplicity. Now we are able to write the conditions on the boundary of two cells in a simple matrix form

$$\mathbf{M}_j \mathbf{B}_j + \mathbf{C}_j = \overline{\mathbf{M}}_{j+1} \mathbf{B}_{j+1} + \overline{\mathbf{C}}_{j+1}. \tag{23}$$

The boundary conditions (9), (10a) and (11) we write in similar form

$$\mathbf{A}_j \mathbf{B}_j + \mathbf{D}_j = 0, \tag{24}$$

$$\mathbf{N}_1 \mathbf{B}_1 + \mathbf{F}_1 = 0, \tag{25}$$

$$\mathbf{N}_J \mathbf{B}_J + \mathbf{F}_J = 0, \tag{26}$$

where

$$\mathbf{A}_j = \begin{bmatrix} f_{10}^{(j)}, & f_{11}, \ldots, & f_{1K} \\ f_{20}, & f_{21}, \ldots, & f_{2K} \\ \cdots & \cdots & \cdots \\ f_{L0}, & f_{L1}, \ldots, & f_{LK} \end{bmatrix}, \qquad \mathbf{N}_1 = \begin{bmatrix} a_{10}^{(1)}, & a_{11}, \ldots, & a_{1K} \\ a_{20}, & a_{31}, \ldots, & a_{2K} \\ \cdots & \cdots & \cdots \\ a_{N_10}, & a_{N_11}, \ldots, & a_{N_1K} \end{bmatrix}, \tag{27}$$

$$\mathbf{N}_J = \begin{bmatrix} c_{10}^{(J)}, & c_{11}, \ldots, & c_{1K} \\ c_{20}, & c_{21}, \ldots, & c_{2K} \\ \cdots & \cdots & \cdots \\ c_{N_10}, & c_{N_11}, \ldots, & c_{N_1K} \end{bmatrix}, \qquad \mathbf{D}_j = \begin{bmatrix} g_1^{(j)} \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_L \end{bmatrix}, \qquad \mathbf{F}_1 = \begin{bmatrix} b_1^{(1)} \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{N_1} \end{bmatrix}, \qquad \mathbf{F}_J \doteq \begin{bmatrix} d_1^{(J)} \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_{N_1} \end{bmatrix}, \tag{28}$$

and where $L = 2N_2 - 1$.

3. SOLUTION OF THE BOUNDARY PROBLEM

The system of equations, describing the boundary conditions for J cells may be written summarily thus:

$$\mathbf{M}\,\mathbf{B} = \mathbf{F}, \tag{29}$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \\ \mathbf{B}_J \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{N}_1 & & & \\ \mathbf{A}_1 & & & \\ \overline{\mathbf{M}}_1, -\overline{\mathbf{M}}_2 & & & \\ & \mathbf{A}_2 & & \\ & \mathbf{M}_2, & -\overline{\mathbf{M}}_3 & \\ & & \mathbf{A}_3 & \\ & & \vdots & \\ & & \mathbf{M}_{J-1}, \overline{\mathbf{M}}_J & \\ & & & \mathbf{A}_J \\ & & & \mathbf{N}_J \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -\mathbf{F}_1 \\ -\mathbf{D}_1 \\ \overline{\mathbf{C}}_2 - \mathbf{C}_1 \\ -\mathbf{D}_2 \\ \overline{\mathbf{C}}_3 - \mathbf{C}_2 \\ -\mathbf{D}_3 \\ \vdots \\ \overline{\mathbf{C}}_J - \mathbf{C}_{J-1} \\ -\mathbf{D}_J \\ -\mathbf{F}_J \end{bmatrix}.$$

By solving the system of equations, represented by equation (29) we obtain directly all the integration constants.

4. DETERMINATION OF THE FLOW RATE THROUGH A CELL

Based on the known integration constants we are able to determine the velocity field in a cell by means of equation (7). The volume flow rate of the medium through a cell is given by the integral (see Fig. 3)

$$G = 2\int_0^{\pi}\int_1^{\bar{\rho}} W(\rho, \varphi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi, \tag{30}$$

where

$$\bar{\rho} = \begin{cases} \dfrac{s}{\cos\varphi} \text{ for } 0 \leqslant \varphi \leqslant \pi/4; & \text{(31a)} \\[2ex] \dfrac{s}{\sin\varphi} \text{ for } \pi/4 < \varphi \leqslant \pi/2; & \text{(31b)} \\[2ex] \dfrac{s}{\sin\varphi} \text{ for } \pi/2 < \varphi \leqslant \pi/2 + \arctan \bar{s}/s; & \text{(31c)} \\[2ex] -\dfrac{\bar{s}}{\cos\varphi} \text{ for } \pi/2 + \arctan \bar{s}/s < \varphi \leqslant \pi. & \text{(31d)} \end{cases}$$

The integration with respect to the variable ρ may be carried out analytically. For the volume flow rate we obtain a simple integral in the form

$$G = 2\int_0^{\pi} [\bar{A}_0(\bar{\rho}) + \sum_{n=0}^{N_3-1} B_n A_n(\bar{\rho}) \cos n\varphi]\,\mathrm{d}\varphi, \tag{32}$$

where

$$\bar{A}_0(\bar{\rho}) = \tfrac{1}{16}(2\bar{\rho}^2 - \bar{\rho}^4)$$

$$A_0(\bar{\rho}) = \tfrac{1}{4}\bar{\rho}^2(2\ln\bar{\rho} - 1)$$

$$A_1(\bar{\rho}) = \tfrac{1}{3}(\bar{\rho}^3 - \bar{\rho} + 2)$$

$$A_2(\bar{\rho}) = \tfrac{1}{4}(\tfrac{1}{4}\bar{\rho}^4 - \ln\bar{\rho} - 1)$$

$$A_n(\bar{\rho}) = \frac{\bar{\rho}^{n+2} - 1}{n+2} - \frac{\bar{\rho}^{-n+2} - 1}{-n+2}$$

$$(n = 3, 4, \ldots, N_3 - 1).$$

The integral (31) has been evaluated numerically by the trapezoidal rule.

5. RESULTS OF THE CALCULATION

The dimensionless pitch selected for this calculation was in the range of $s = 1{\cdot}50$, 1·75, 2·00, 2·25, 2·50. Lower values of s were not considered, since for $s = 1{\cdot}5$ the influence of the wall on the second cell row is already negligible. Values larger than $s = 2{\cdot}5$ will hardly occur in practice. The wall distance has been varied from $s_1 = s$ (the first cell is square) to a value corresponding to the 1·3 times of the hydraulic diameter of the cell in the bundle.

The calculation has been carried out for thirteen boundary points ($N_1 = N_2 = 4$). Then according to (20) we obtain for a number of terms in the expansion N_3 a value of 15. In the numerical integration of expression (32) twenty mesh points have been selected in the individual sections (31).

Figures 4 and 5 show the volume flow rates in non-influenced cells. Figure 4 illustrates the flow rate G_1 through the first cell as a function of the dimensionless pitch and the distance of the fixed wall from the axes of the first row of cylinders s_1. In the figure a line is also plotted, corresponding to the case where the first cell is square ($s_1 = s$), and the line, which corresponds to a case where the hydraulic diameter of the first cell d_1 is the same as the hydraulic diameter of the cell within the bundle d_i, i.e.

$$d_1 = d_i$$

where

$$d_1 = \frac{4s(s + s_1) - 2\pi}{\pi + s}, \tag{33}$$

$$d_i = \frac{8s^2}{\pi} - 2. \tag{34}$$

The flow rate G_i through the non-influenced cell within the bundle is plotted in Fig. 5. The boundary condition for the surface of this cell takes the form of:

$$\frac{\partial W}{\partial v} = 0,$$

where v denotes the external normal to the surface. Results for G_i agree with paper [2].

The ratio

$$\eta = G/G_h$$

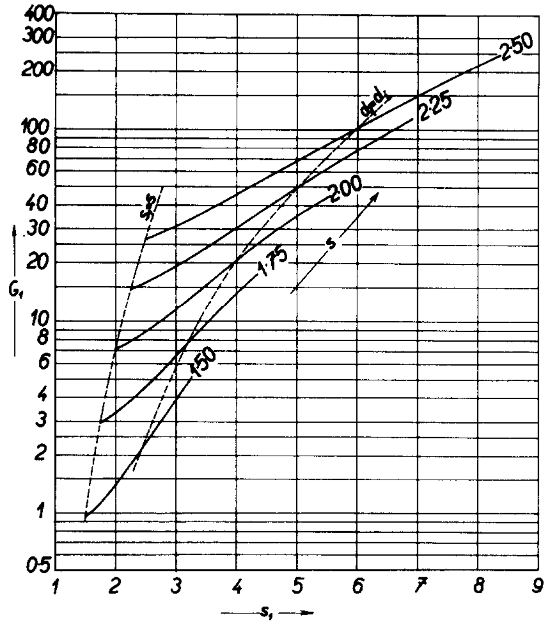


FIG. 4. The dimensionless volume flow rate G_1 in the non-influenced first cell against the values of s and s_1 (the broken line $s_1 = s$ corresponds to the case where the first cell is square, the broken line $d_1 = d_i$ corresponds to the case where the hydraulic diameter of the first cell is equal to that of the cell within the bundle).

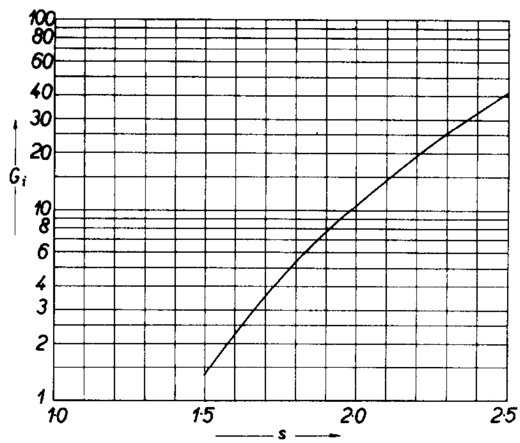


FIG. 5. The dimensionless volume flow rate G_i in the non-influenced cell within the bundle against the value of s.

has been selected as a characteristic value for the mutual influence of the cells. G is the flow rate through the influenced cell in the bundle and G_h is the flow rate through the non-influenced cell, i.e. $G_h = G_1$ for the first cell and $G_h = G_i$ for the cell within the bundle. Figure 6 and 7 show plots of the value η against the pitch and the ratio of hydraulic diameters

$$\xi = \frac{d_1}{d_i}$$

for the first and second row of cells.

The resulting values of η for various values of s and ξ are given in Figs. 8(a)–(e). The independent variable is the serial number of the cell row. It is found that the influence on the third row is already very slight. The fourth row is not influenced at all in the given range of s and ξ values. Numerical calculations for small values of s were carried out with $J = 3$ and for larger s with $J = 4$.

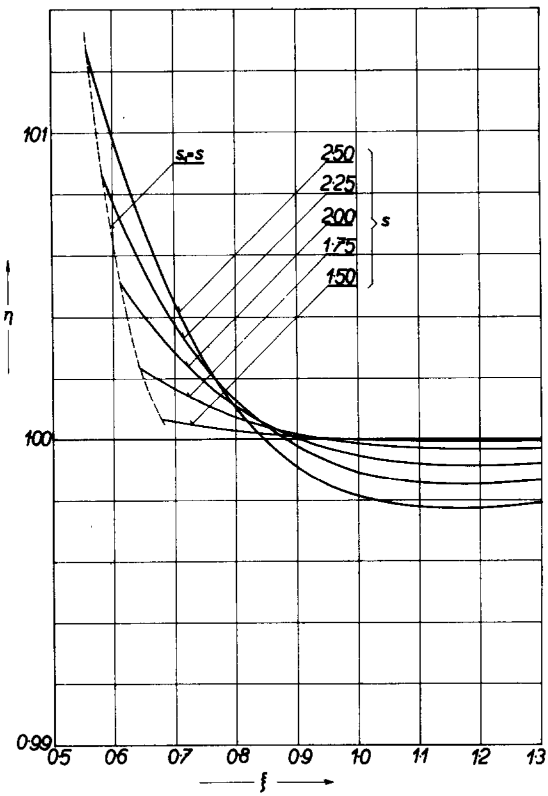


FIG. 6. Plot of η for the first cell row of the bundle against s and ξ (the broken line $s_1 = s$ corresponds to the case where the first cell is square).

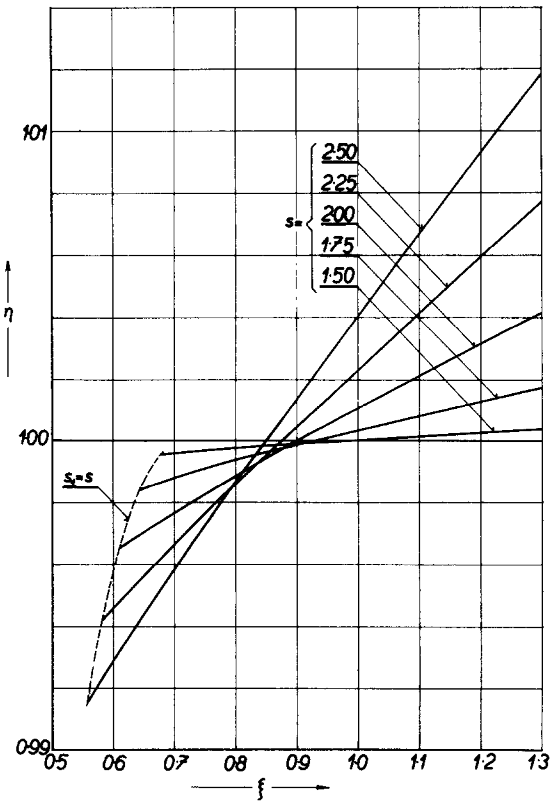


FIG. 7. Plot of η for the second cell row against s and ξ (the broken line $s_1 = s$ corresponds to the case where the first cell is square).

It must be remarked, however, that the influence exerted upon the individual cell rows is very small from a practical point of view, and exceeds 1 per cent only for pitches of $s > 2·5$ and $\xi > 1·3$ or $\xi < 0·6$. For illustration, the velocity field in the first row of cells is plotted in Fig. 9 for $s = 1·5$ and $s_1 = 3·0$.

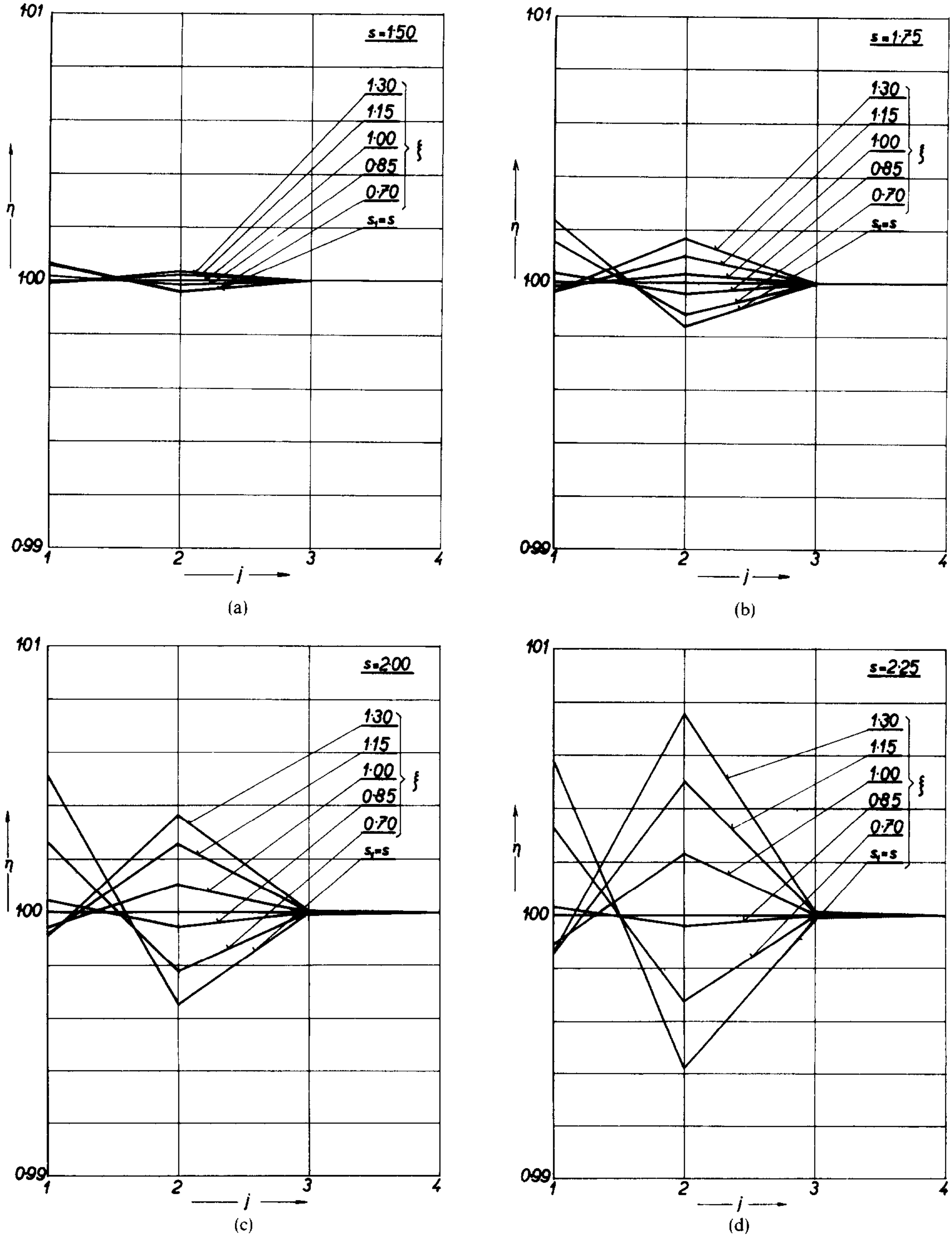
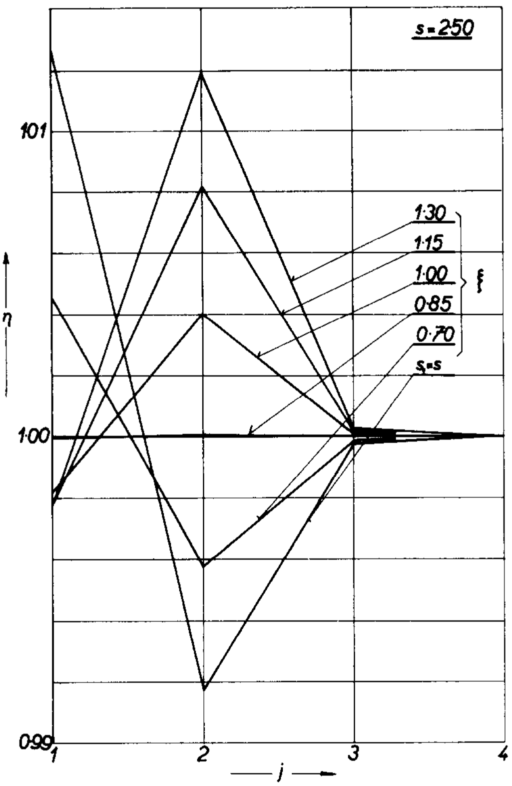


FIG. 8. Plot of η against the serial number of the row for individual values of ξ and s: (a) $s = 1{\cdot}50$, (b) $s = 1{\cdot}75$, (c) $s = 2{\cdot}00$, (d) $s = 2{\cdot}25$, (e) $s = 2{\cdot}50$.



(e)

FIG. 8—*continued.*

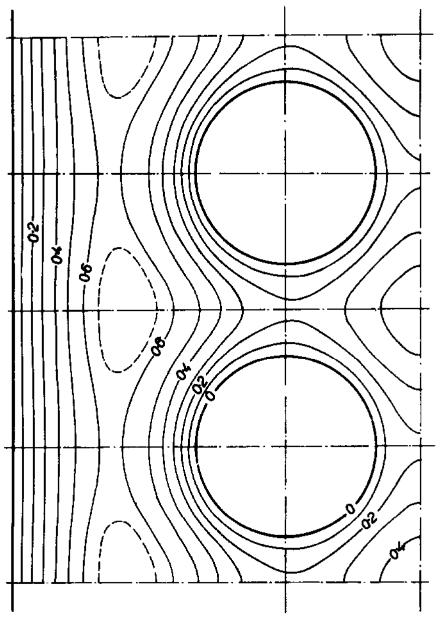


FIG. 9. The velocity field in the non-influenced cell row for $s = 1{\cdot}5$, $s_1 = 3{\cdot}0$.

6. ESTIMATE OF THE PRECISION OF THE CALCULATION

6.1. *Influence of the number of terms in the expansion* (7) *on the flow rate*

To determine the accuracy of the solution depending on the number of boundary points, the problem was solved for the first cell for $s = 2{\cdot}5$ and for various numbers $N = N_1 = N_2 = 3, 4, 5$. To these numbers of boundary points correspond 11, 15, 19 terms respectively in the expansion (7). The flow rate values $G^{(3)}$, $G^{(4)}$, $G^{(5)}$ were then used to calculate, by Aitken's method, the asymptotic values for $N \to \infty$ according to the relation

$$G^{as} = G^{(3)} + \frac{\delta^{4,3}}{1 - \dfrac{\delta^{5,4}}{\delta^{4,3}}},$$

where

$$\delta^{i,j} = G^{(i)} - G^{(j)}.$$

Figure 10 represents the ratio of the flow rate x

$$x = \frac{G^{(N)}}{G^{as}}$$

against ξ for $N = 3, 4, 5$. It is found, that up to $\xi = 1{\cdot}0$ the flow rate values for $N = 4$ and

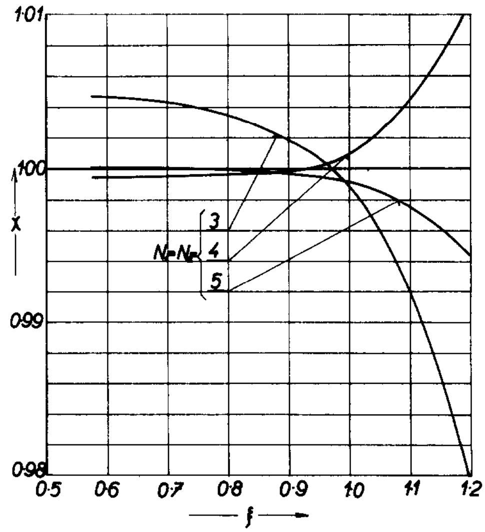


FIG. 10. Comparison of x for various numbers of boundary points.

$N = 5$ are very close to the estimated correct solution. In the interval $1{\cdot}0 < \xi < 1{\cdot}15$ the precision is acceptable for the case solved ($N = 4$).

Figure 11 gives values of η for different values of N. We find that for $\xi < 1{\cdot}15$ the resulting values of η for $N = 4$ and $N = 5$ are identical. The solution for $N = 3$ shows a larger deviation

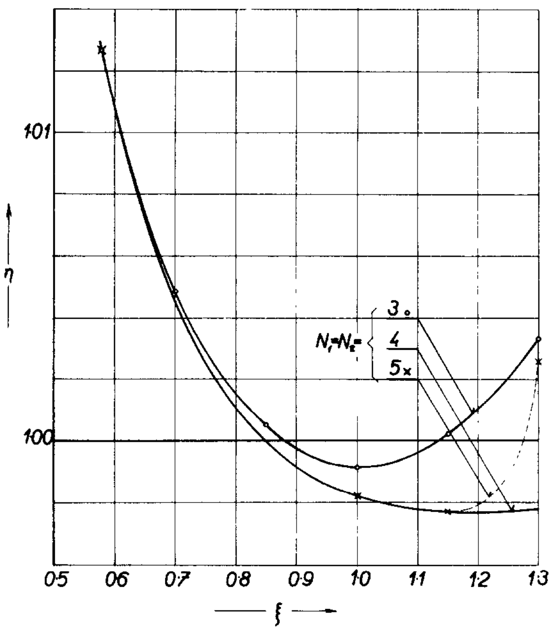


FIG. 11. Comparison of η for various numbers of boundary points.

in the whole ξ range. The value of η for $\xi = 1{\cdot}3$ and $N = 5$ is, contrary to the expected result, incorrect. This is due to the fact that we are evidently dealing with the problem of a correct representation of the number in a computer. In the equations for the integration constants the values of $\bar{\rho}_{\min}^{N_3-1}$ and $\bar{\rho}_{\max}^{N_3-1}$ are combined (see Fig. 3), their ratio being in this case about $\sim 10^{10}$, which is a value about two orders of magnitude higher, than the precision of representation of the number in the computer used ($\sim 5\,.\,10^{7}$). From this point of view, the results in the range of $\xi > 1{\cdot}15$ and $N = 4$ may be considered as correct.

6.2. *Influence of the number of mesh points in numerical integration, on the flow rate*

As already mentioned in section 4, the integral (32) has been evaluated numerically for every case, using the trapezoidal rule. To verify the precision, the dependence of the flow rate on the number of mesh points has been determined for one case. The results are given in Table 1 [n denotes the number of mesh points in every section (31)].

Table 1

n	G
2	24·5874
4	25·4177
5	25·5218
6	25·5788
8	25·6357
10	25·6621
16	25·6908
20	25·6974

From the results obtained for $n = 2, 4, 8, 16$ or $n = 5, 10, 20$ it is possible to estimate the asymptotic flow rate value, using Aitken's method. It has been found that for the twenty mesh points used in our computation the error is less than 0·05 per cent.

7. CONCLUSION

The influence of the wall on laminar flow in a bundle of circular cylinders has been evaluated. It has been found that, for cases which come to be considered from a practical point of view, the mutual influence of the flow rates of the cells is small. The influence in the first and second row of cells will exceed a value of 1 per cent only for $s > 2{\cdot}5$, $\xi > 1{\cdot}3$ and $s > 2{\cdot}5$, $\xi < 0{\cdot}6$. The influence of the wall on the third row of cells is negligible.

8. ACKNOWLEDGEMENT

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Résumé—Le travail décrit les recherches théoretiques de l'écoulement développé laminaire dans un réseau carré infini des cylindres circulaires, ce réseau étant limité à côte par un paroi parallèle aux axes des cylindres. On a étudié l'effet du pas du réseau ($2b$) et de écartement du paroi de la première ligne des cylindres (b_1) sur l'écoulement de fluide par les éléments particulaires du réseau. La quantité b était variée dans une gamme de 1,5 R à 2,5 R (R est le radius du cylindre) et l'écartement du paroi était varié de $b_1 = b$ (le premier élément est carré) aux telles valeurs de b_1 que le rapport des diamètres hydrauliques du premier éléments et de l'étément à l'intèrieur du réseau ($\xi = d_1/d_i$) atteignait la valeur de 1,3. Le calcul a donné que l'effet mutuel des écoulements dans les éléments particuliers est très faible. Les variations de l'écoulement des premiers deux lignes d'éléments n'atteignent la valeur de 1% que pour les pas relativement importants ($b > 2{,}5R$) et pour les cas quand le premier élément diffère considérablement de l'élément à l'intérieur du réseau ($\xi > 1{,}3$ et $\xi < 0{,}6$). L'ettet de la troisième lignes des éléments est négligible dans la gamme des paramètres étudiés.

Zusammenfassung—Die Arbeit beschreibt eine theoretische Untersuchung der ausgebildeten, laminaren Längsströmung um Kreiszylinder in einer unendlichen Rechteckanordunung. Die Anordnung wird auf einer Seite von einer zu den Zylinderachsen parallelen Wand begrenzt. Der Einfluss der Anordnungsteilung ($2b$) und der Abstand der Wand von der ersten Zylinderreihe (b_1) auf die Strömungsgeschwindigkeit des Mediums in den einzelen Zellen der Anordnung wird untersucht und die gegenseitige Beeinflussung der Geschwindigkeiten in den Einzelzellen ermittelt. Die Grösse b wirde im Bereich von 1,5 R bis 2,5 R (R ist der Zylinderradius) verändert und der Wandabstand von $b_1 = b$ (die erste Zelle ist quadratisch) bis auf solche Werte von b_1 variiert, für die das Verhältnis des hydraulischen Durchmessers der ersten Zelle zu dem einer Zelle innerhalb des Gitters ($\xi = d_1/d_i$) den Wert 1,3 erreicht. Aus der Berechnung folgt, dass der gegenseitige Einfluss der einzelen Zellen aufeinander relativ gering ist. Geschwindigkeitsänderungen in den ersten beiden Reihen erreichen einen Wert von 1% nur bei verhältnismässig grossen Teilungen ($b >$ 2,5 R) und bei grossen Unterschieden zwischen dem hydraulischen Durchmesser der ersten Zelle und jenem der Zellen im Gitter ($\xi > 1{,}3$ und $\xi < 0{,}6$). Der Einfluss der dritten Reihe ist vernachlässigbar im Bereich der betrachteten Parameter.

Аннотация—В статье описывается теоретическое исследование продольного развитого ламинарного течения в бесконечной квадратной решетке круговых цилиндров, ограниченной с одной стороны стенкой, параллельной осям цилиндров. Исследуется влияние расстояния между осями цилиндров ($2b$) и расстояния от стенки до первого ряда цилиндров (b_1) на скорость течения среды через отдельные элементы решетки, а также взаимное влияние скоростей течения в элементах решетки. Величина b изменялась от 1,5 R до 2,5 R /где R—радиус цилиндра/, а расстояние от стенки от $b_1 = b$/ первая ячейка—квадрат/ до таких значений b_1, когда отношение гидравлических диаметров первой ячейки и ячейки внутри решетки /$\xi = d_1/d_i$/ достигало 1,3. Расчеты показывают, что взаимное влияние отдельных ячеек довольно незначительно. Изменения скорости до 1% в первых двух рядах наблюдалось только при относительно больших расстояниях между осями цилиндров /$b > 2{,}5$ R/, а также в случаях значительной разности гидравлических диаметров цилиндров первого ряда и внутри решетки /$\xi > 1{,}3$ u $\xi < 0{,}6$/. В исследуемом диапазоне параметров влиянием третьего ряда цилиндров можно пренебречь.